Exercise 21

The force F acting on a body with mass m and velocity v is the rate of change of momentum: $F = (d/dt)(mv)$. If m is constant, this becomes $F = ma$, where $a = dv/dt$ is the acceleration. But in the theory of relativity the mass of a particle varies with v as follows: $m = m_0/\sqrt{1 - v^2/c^2}$, where m_0 is the mass of the particle at rest and c is the speed of light. Show that

$$F = \frac{m_0 a}{(1 - v^2/c^2)^{3/2}}$$

Solution

Calculate F by using the chain rule and the quotient rule.

$$\begin{aligned}
F &= \frac{d}{dt}(mv) \\
&= \frac{dv}{dt}\frac{d}{dv}(mv) \\
&= a\frac{d}{dv}\left(\frac{m_0 v}{\sqrt{1 - v^2/c^2}}\right) \\
&= m_0 a\frac{d}{dv}\left(\frac{v}{\sqrt{1 - v^2/c^2}}\right) \\
&= m_0 a\frac{\left[\frac{d}{dv}(v)\right]\sqrt{1 - v^2/c^2} - \left[\frac{d}{dv}\left(\sqrt{1 - v^2/c^2}\right)\right](v)}{1 - v^2/c^2} \\
&= m_0 a\frac{(1)\sqrt{1 - v^2/c^2} - \left[\frac{1}{2}(1 - v^2/c^2)^{-1/2} \cdot \frac{d}{dv}(1 - v^2/c^2)\right](v)}{1 - v^2/c^2} \\
&= m_0 a\frac{\sqrt{1 - v^2/c^2} - \left[\frac{1}{2}(1 - v^2/c^2)^{-1/2} \cdot \left(-\frac{2v}{c^2}\right)\right](v)}{1 - v^2/c^2} \\
&= m_0 a\frac{\sqrt{1 - v^2/c^2} + \frac{v^2}{c^2}\frac{1}{\sqrt{1 - v^2/c^2}}}{1 - v^2/c^2} \\
&= m_0 a\frac{\frac{(1-v^2/c^2)+v^2/c^2}{\sqrt{1-v^2/c^2}}}{1 - v^2/c^2} \\
&= \frac{m_0 a}{(1 - v^2/c^2)^{3/2}}
\end{aligned}$$